Solution Set, Math 232 Midterm 2, Version 1 (Green/Grey)

Spring 2012, Simon Fraser University

1(a).

$$\begin{aligned}\mathbf{u}\times\mathbf{v} &= \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 1 & 5 & -2\\ -2 & 0 & 3\end{vmatrix}\\ &= 15\hat{\mathbf{i}}+\hat{\mathbf{j}}+10\hat{\mathbf{k}}\end{aligned}$$

1(b).

$$\begin{aligned}(\mathbf{u}\times\mathbf{u})\times\mathbf{v} &= \mathbf{0}\times\mathbf{v}\\ &= \mathbf{0}\end{aligned}$$

where the first step uses the fact that cross product of two parallel vectors is the zero vector. You can also get the right answer by calculating directly.

1(c).

$$\begin{aligned}\mathbf{u}\times(\mathbf{v}\times\mathbf{u}) &= \mathbf{u}\times(-\mathbf{u}\times\mathbf{v})\\ &= (1,5,-2)\times(-15,-1,-10)\\ &= \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 1 & 5 & -2\\ -15 & -1 & -10\end{vmatrix}\\ &= -52\hat{\mathbf{i}}+40\hat{\mathbf{j}}+74\hat{\mathbf{k}}\end{aligned}$$

(We have used the fact that $\mathbf{a}\times\mathbf{b}=-\mathbf{b}\times\mathbf{a}$. We then use the value of $\mathbf{u}\times\mathbf{v}$ from part (a). You can also get the correct answer by calculating directly.)

1(d). The area of the parallelogram is $||\mathbf{u} \times \mathbf{v}||$. We use the value of $\mathbf{u} \times \mathbf{v}$ from part (a) to see that this is $||(15, 1, 10)|| = \sqrt{15^2 + 1^2 + 10^2} = \sqrt{326}$.

2(a).

$$\begin{aligned}\frac{3+2i}{1+4i} &= \left(\frac{3+2i}{1+4i}\right)\left(\frac{1-4i}{1-4i}\right) \\ &= \frac{3-12i+2i-8i^2}{1-16i^2} \\ &= \frac{11-10i}{17} \\ &= \frac{11}{17} + \left(-\frac{10}{17}\right)i\end{aligned}$$

2(b). Note that $|\sqrt{3}+i| = \sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{3+1} = 2$, so we can write

$$\sqrt{3} + i = 2(\cos\theta + i\sin\theta)$$

where $\cos\theta = \sqrt{3}/2$ and $\sin\theta = 1/2$, so $\theta = \pi/6$. Then

$$\begin{aligned}(\sqrt{3}+i)^7 &= 2^7(\cos(7\theta) + i\sin(7\theta)) \\ &= 128(\cos(7\pi/6) + i\sin(7\pi/6)) \\ &= 128\left(\frac{-\sqrt{3}}{2} - \frac{i}{2}\right) \\ &= -64\sqrt{3} + (-64)i.\end{aligned}$$

2(c).

$$\begin{aligned}\det(A) &= \begin{vmatrix} 1 & 1 & 3 & 0 \\ 0 & 4 & 1 & 1 \\ 0 & -3 & 2 & 1 \\ 0 & 1 & -1 & 4 \end{vmatrix} \\ &= \begin{vmatrix} 4 & 1 & 1 \\ -3 & 2 & 1 \\ 1 & -1 & 4 \end{vmatrix} \\ &= 32 + 1 + 3 + 4 + 12 - 2 \\ &= 50.\end{aligned}$$

(We use a row operation—add two times the first row to the second row—to simplify the determinant in the first step, and then cofactor expansion on the first column in the second step. There are many other valid approaches.)

3(a). $A = \begin{pmatrix} 7/8 & 1/2 \\ 1/8 & 1/2 \end{pmatrix}$ (Where the ground state corresponds to the first row and first column and the excited state corresponds to the second row and second column.)

3(b). The state at time $t = 0$ is $\mathbf{x}(0) = \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}$, so the state at time $t = 1$ is

$$\begin{aligned} \mathbf{x}(1) &= A\mathbf{x}(0) \\ &= \begin{pmatrix} 7/8 & 1/2 \\ 1/8 & 1/2 \end{pmatrix} \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix} \\ &= \begin{pmatrix} 3/4 \\ 1/4 \end{pmatrix}, \end{aligned}$$

so $1/4$ of the molecules are in the excited state at time $t = 1$. (Note that you do not actually need to calculate the first component of $\mathbf{x}(1)$ to get the answer: calculating the second component suffices.)

3(c). The limit must be a fixed point of A and also a probability vector. So solve $A\mathbf{x} = \mathbf{x}$, or equivalently $(I - A)\mathbf{x} = \mathbf{0}$. This has augmented matrix form

$$\left(\begin{array}{cc|c} 1/8 & -1/2 & 0 \\ -1/8 & 1/2 & 0 \end{array}\right)$$

and we can add the first row to the second to obtain

$$\left(\begin{array}{cc|c} 1/8 & -1/2 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

and scale the first row by 8 to obtain

$$\left(\begin{array}{cc|c} 1 & -4 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

So the solutions are of the form

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4t \\ t \end{pmatrix}$$

for $-\infty < t < \infty$. These are the fixed points of A. We need to find one that is a probability vector, that is, with $4t + t = 1$, so that $t = 1/5$. So the limit is

$$\begin{pmatrix} 4/5 \\ 1/5 \end{pmatrix},$$

so that, in the limit as $t \to \infty$, we have $1/5$ of the molecules are in the excited state.

4(a). The domain and codomain are both $\mathbb{R}^3$

4(b). $[T] = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

4(c).

$$\begin{aligned} T(2,1,-5) &= [T] \begin{pmatrix} 2 \\ 1 \\ -5 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ -5 \end{pmatrix} \\ &= \begin{pmatrix} 7 \\ -3 \\ 1 \end{pmatrix} \end{aligned}$$

4(d). This is the same as finding a vector $\mathbf{x} = (x_1, x_2, x_3)$ such that

$$[T] \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}.$$

Which is the same as solving the linear system with augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \end{array}\right)$$

which we can row-reduce by subtracting the first row from the second

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 0 & 0 & 2 & -2 \\ 0 & 1 & 0 & 2 \end{array}\right)$$

then scale the second row by $1/2$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 2 \end{array}\right)$$

then add the second row to the first

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 2 \end{array}\right).$$

So the solution is $(x_1, x_2, x_3) = (2, 2, -1)$.

4(e). It is not an isometry because it does not preserve length: $T(1,0,0) = (1,1,0)$, but $||(1,0,0)|| = \sqrt{1^2+0^2+0^2} = 1$ while we have $||(1,1,0)|| = \sqrt{1^2+1^2+0^2} = \sqrt{2}$. (You can also use $T(0,0,1) = (-1,1,0)$ to see that T is not length-preserving, or check that $[T]$ is not an orthogonal matrix.)

5(a). Since $\begin{pmatrix}1\\1\\1\end{pmatrix}$ is an eigenvector with eigenvalue $\lambda = -1$, we have

$$\begin{aligned} A^5 \begin{pmatrix}1\\1\\1\end{pmatrix} &= \lambda^5 \begin{pmatrix}1\\1\\1\end{pmatrix} \\ &= (-1)^5 \begin{pmatrix}1\\1\\1\end{pmatrix} \\ &= \begin{pmatrix}-1\\-1\\-1\end{pmatrix}. \end{aligned}$$

5(b). Since $\begin{pmatrix}1\\1\\1\end{pmatrix}$ is an eigenvector with eigenvalue -1, we have

$$A \begin{pmatrix}1\\1\\1\end{pmatrix} = -1 \begin{pmatrix}1\\1\\1\end{pmatrix}$$

so that

$$\begin{pmatrix} a & b & c \\ -3 & d & 3 \\ -9 & 0 & e \end{pmatrix} \begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}-1\\-1\\-1\end{pmatrix},$$

that is

$$
\begin{aligned}
a + b + c &= -1 \\
-3 + d + 3 &= -1 \\
-9 + 0 + e &= -1,
\end{aligned}
$$

so that $c = -1 - a - b$, $d = -1$, $e = 8$, so

$$A = \begin{pmatrix} a & b & -1 - a - b \\ -3 & -1 & 3 \\ -9 & 0 & 8 \end{pmatrix}.$$

The trace of a matrix is the sum of its eigenvalues (repeated according to multiplicity), so $\mathrm{tr}(A) = 2 + (-1) + (-1) = 0$. The trace is also the sum of the diagonal elements, so $a + (-1) + 8 = 0$, and so $a = -7$. Thus

$$A = \begin{pmatrix} -7 & b & 6 - b \\ -3 & -1 & 3 \\ -9 & 0 & 8 \end{pmatrix}.$$

The determinant of a matrix is the product of its eigenvalues (repeated according to multiplicity), so $\det(A) = (2)(-1)(-1) = 2$, but we can also compute $\det(A)$ directly from our expression for A

$$\det(A) = 56 - 27b + 0 - 0 + 24b - 9(6 - b) = 2 + 6b.$$

So since $\det(A) = 2$, this means $b = 0$ and so

$$A = \begin{pmatrix} -7 & 0 & 6 \\ -3 & -1 & 3 \\ -9 & 0 & 8 \end{pmatrix}.$$

So $a = -7$, $b = 0$, $c = 6$, $d = -1$, and $e = 8$.